ON THE SUMMING OF $\sum_{n=1}^{\infty} n^m x^n$

In a recent article we have shown that-

$$m! = \lim_{z \to 1} \frac{1}{z} \left\{ (1-z)^{m+1} \sum_{n=0}^{\infty} n^m z^n \right\} = \sum_{n=1}^{m} D[n, m] \quad \text{for fixed} \quad m$$

Here $D(n,m)$ represents the coefficients of a new type of Pascal triangle given by-

$$D[n, m] = \sum_{k=1}^{m} (-1)^{k-1} \frac{(m+1)! (n+1-k)^m}{(k-1)! (m+2-k)!}$$

If we let the sum in the first equality be $S[z,m]$ we have-

$$S[z,m] = \sum_{n=1}^{\infty} n^m z^n = z + 2^m z^2 + 3^m z^3 + ...$$

At first glance this series seems to not be sum able. However if we write out the first few values for fixed integer $m$ we obtain-

$$S[z,1] = \frac{z[1]}{(1-z)^2}$$
$$S[z,2] = -\frac{z[1+z]}{(1-z)^3}$$
$$S[z,3] = \frac{z[1+4z+z^2]}{(1-z)^4}$$
$$S[z,4] = -\frac{z[1+11z+11z^2+z^3]}{(1-z)^5}$$

This has a clear pattern with the denominator going as $(1-z)^{m+1}$ and the sign in front of $z$ in the numerator going as $(-1)^{m+1}$. The $m-1$ order polynomials inside the square bracket have the coefficients $D[n,m]$. Thus we arrive at a closed form solution-

$$S[z,m] = \frac{z\{D[1, m] + D[2, m]z + ... + D[m, m]z^{m-1}\}}{(1-z)^{m+1}}$$

We can pick out the values of the $D[n,m]$s by using the above formula for these coefficients or even easier by just looking at the modified Pascal Triangle-
It shows us, for example at $z=1/2$ and $m=5$, that-

$$S_{1/2,5} = \sum_{n=1}^{\infty} n^5 \left(1/2\right)^n = \lim_{z \to 1/2} \{z\{1 + 26z + 66z^2 + 26z^3 + z^4\}/(z - 1)^6\} = 1082$$

What this shows is that we are able to evaluate infinite sums of the type $S[z,m]$ using the sum of just $m$ terms provided that $m$ remains a positive integer and $0 < z < 1$. The sum will always blowup at $z=1$ and fails to converge for $z>1$.

It is possible for $z$ to be complex so that $z=\sigma+i\tau$, provided the ratio test for convergence is satisfied. For finite positive integer $m$ this means $|z|<1$. Thus we have for $m=3, \sigma=0$, and $\tau=1/2$ that-

$$\sum_{n=1}^{\infty} n^3 \left(i/2\right)^n = \lim_{z \to i/2} \{z(1 + 4z + z^2)\}/(z - 1)^4 = (-1/625)(32 + 426i)$$

We can even handle-

$$\sum_{n=1}^{\infty} \frac{n^3}{2^{n-1}} \cos\left(\frac{\pi}{2} n\right) = -2(32)/625 = -0.1024000$$

recalling the identities-

$$\cos\left(\frac{n\pi}{2}\right) = -\left[\frac{i^n + i^{-n}}{2}\right] \quad \text{and} \quad \sin\left(\frac{n\pi}{2}\right) = \left[\frac{i^n - i^{-n}}{2i}\right]$$